Axioms of Set Theory

1. Zermelo-Frankel Set Theory (ZF)

Axiom (Extension)**.** *Two sets are equal if and only if they have the same elements.*

Axiom (Specification)**.** *To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

This axiom is how we create subsets. We are picking out the elements x which make the conditional statement $S(x)$ true.

Axiom (Pairing)**.** *Given sets A and B, the set $\{A, B\}$ exists.*

Note that this is different from taking a union. This is simply saying that if we have some things, we can make a set with those things in it. In a way, this is saying that we can take an empty box $\{\}$ and put things inside of it to make new sets. The reason that A and B are sets here is because in an axiomatic system, we only have the things given to us by its axioms. The only things we get are sets, so this means that everything has to be constructed from sets.

Axiom (Unions)**.** *For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.*

Axiom (Powers)**.** *For each set there exists a collection of sets that contains among its elements all the subsets of the given set.*

This axiom allows us to take the power set of a set.

Axiom (Infinity)**.** *There exists a set S such that $\varnothing \in S$ and for every $X \in S$, the set $X^+ := X \cup \{X\} \in S$.*

The set X^+ is called the successor of X. This is an interesting axiom. At its most basic level, it is just saying there is an infinite set. This set it gives us is actually $\mathbb{N}$. It does it in the following way

- $0 := \varnothing$
- $1 := 0^+ = \varnothing \cup \{\varnothing\} = \{\varnothing\}$
- $2 := 1^+ = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$
- $3 := 2^+ = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

etc. One also says that this axiom gives us the existence of an *inductive* set.

Axiom (Substitution)**.** *If $S(a, b)$ is a sentence such that for each a in a set A the set $\{b | S(a, b)\}$ can be formed, then there exists a function F with domain A such that $F(a) = \{b | S(a, b)\}$ for each $a \in A$.*

The best way to explain this is with an example. Suppose A is the set $\mathbb{N}$ and the sentence $S(a, b)$ is "b is divisible by a," then the set $F(a)$ is the set of all numbers $b \in \mathbb{N}$ such that a divides b. For example

$$F(1) = \mathbb{N}$$
$$F(2) = \{2, 4, 6, 8, 10, 12, 14, ...\}$$
$$F(3) = \{3, 6, 9, 12, 15, 18, 21, ...\}$$

etc. This is somewhat related to creating indexed sets, but has a much larger scope.

Axiom (Regularity). *For every non-empty set A there is an element $X \in A$ such that $X \cap A = \varnothing$.*

This may seem like a very strange thing to require, but this axiom prevents issues like *Russell's paradox.* This axiom is mainly to prevent logical paradoxes like that and doesn't much show up otherwise.

2. Axiom of Choice

The 8 axioms above comprise what is known as Zermelo-Frankel Set Theory, which we call ZF. However, ZF is lacking in many ways. One way to start to make up for this to include the *Axiom of Choice.* Adding in this axiom is denoted ZFC.

Axiom (Choice). *The Cartesian product of a non-empty family of non-empty sets is non-empty.*

An element of the Cartesian product would be a tuple with one entry coming from each of the sets in the family, which would be possible since each set is non-empty.

2.1. Equivalent Forms.

Axiom. *For any collection X of non-empty sets, there exists a choice function f.*

A choice function is a function $f : X \to \cup X$ such that $f(A) \in A$.

Axiom. *Given any set X of pairwise disjoint non-empty sets, there exists at least one set C that contains exactly one element in common with each of the sets in X.*

This axiom says that, given a collection of non-empty sets, no two of which have any elements in common, we can make at least one new set, C, by taking one element from every one of the sets in the collection.

3. Other Useful Statements Equivalent to the Axiom of Choice

Axiom (Well-Ordering Theorem). *Every set can be well-ordered.*

A set is said to be well-ordered if every non-empty subset has a smallest element.

Axiom (Zorn's Lemma). *Every non-empty partially ordered set in which every chain (i.e. totally ordered subset) has an upper bound contains at least one maximal element.*